

LINEAR SYSTEMS OF RATIONAL CURVES ON RATIONAL SURFACES

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Dedicated to the memory of Vladimir Igorevich Arnold

Let S be a projective nonsingular rational surface, over an algebraically closed field of characteristic zero. We say that a linear system \mathbb{L} on S is *rational* if $\dim \mathbb{L} \geq 1$ and the general member of \mathbb{L} is an irreducible rational curve.

Let $C \subset S$ be an irreducible curve.

Let Ω_C be the set of rational linear systems \mathbb{L} on S satisfying $C \in \mathbb{L}$. Consider the minimal¹ resolution of singularities $\pi : \tilde{S} \rightarrow S$ of C , let \tilde{C} be the strict transform of C on \tilde{S} , and let $\tilde{\nu}(C)$ denote the self-intersection number of \tilde{C} in \tilde{S} . Then Theorem 2.8 implies:

(1) $\Omega_C \neq \emptyset$ if and only if C is rational and $\tilde{\nu}(C) \geq 0$.

Let \mathbb{L}_C be the linear system on S which is the image of $|\tilde{C}|$ by $\pi_* : \text{Div}(\tilde{S}) \rightarrow \text{Div}(S)$ (so $C \in \mathbb{L}_C$). Assuming that $\Omega_C \neq \emptyset$, we show (Thm 2.8):

(2) For any linear system \mathbb{L} on S , $\mathbb{L} \in \Omega_C \iff \dim \mathbb{L} \geq 1$ and $C \in \mathbb{L} \subseteq \mathbb{L}_C$.

This gives a complete description of Ω_C , and we note in particular that Ω_C has a greatest element (namely, \mathbb{L}_C). Continuing to assume that $\Omega_C \neq \emptyset$, Theorem 2.8 also shows that $\dim \mathbb{L}_C = \tilde{\nu}(C) + 1$ and that the minimal resolution of singularities of C coincides with the minimal resolution of the base points of \mathbb{L}_C .

The present paper may be viewed as a preamble to the forthcoming [2], in which we study linear systems associated to unicuspidal rational curves $C \subset \mathbb{P}^2$. We remind the reader that all currently known curves of that type satisfy $\tilde{\nu}(C) \geq 0$, hence $\Omega_C \neq \emptyset$. It is shown in [2] that if $C \subset \mathbb{P}^2$ is a unicuspidal rational curve with singular point P then: (1) there exists a unique pencil Λ_C on \mathbb{P}^2 satisfying $C \in \Lambda_C$ and $\text{Bs}(\Lambda_C) = \{P\}$; (2) Λ_C is a rational pencil if and only if $\tilde{\nu}(C) \geq 0$; (3) if $\tilde{\nu}(C) \geq 0$, then Λ_C has a dicritical of degree 1. From these facts, the paper then derives consequences for the classification of unicuspidal rational curves $C \subset \mathbb{P}^2$ satisfying $\tilde{\nu}(C) \geq 0$.

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¹This is the “short” resolution, not the “embedded” resolution. See 1.4 for details.

Conventions. All algebraic varieties are over an algebraically closed field \mathbf{k} of characteristic zero. Varieties are irreducible and reduced, so in particular all curves are irreducible and reduced. A divisor D of a surface is *reduced* if $D = \sum_{i=1}^n C_i$ where C_1, \dots, C_n are distinct curves ($n \geq 0$).

1. CLUSTERS ON A SURFACE

We fix a projective nonsingular surface S throughout this section. We consider the set S^* of points which are either points of S or points infinitely near points of S . The set S^* comes equipped with a partial order \leq , called the *natural order*, such that for $P, Q \in S^*$ we have $P < Q$ if and only if Q is infinitely near P . The minimal elements of (S^*, \leq) are called the *proper points* of S , and are indeed in bijective correspondence with the closed points of S . Note that the poset (S^*, \leq) is a classical object (for instance it is called a “bubble space” in [4] but has the order relation reversed).

A *cluster* on S is a (possibly empty) finite subset $K \subset S^*$ such that, given any $P, Q \in S^*$, if $P \leq Q$ and $Q \in K$ then $P \in K$. If K is a cluster on S then a *subcluster* of K is any subset of K which is itself a cluster on S . Note that if K is a cluster on S then each minimal element of K is a proper point of S .

The aim of this section is to fix the notations and terminologies for clusters and to recall certain facts in that theory—there are no new results here. Our main reference is the first chapter of [1], and our notations and definitions are in general compatible with that text.

1.1. Let K be a cluster on S .

- (a) The *blowing-up of S along K* is denoted $\pi_K : S_K \rightarrow S$. Observe that if K' is a subcluster of K then $K \setminus K'$ is a cluster on $S_{K'}$ and π_K factors as

$$S_K \xrightarrow{\pi_{K \setminus K'}} S_{K'} \xrightarrow{\pi_{K'}} S. \quad (1)$$

- (b) Given a divisor $D \in \text{Div}(S)$, let $\tilde{D}^K \in \text{Div}(S_K)$ and $\overline{D}^K \in \text{Div}(S_K)$ denote, respectively, the strict transform and total transform of D on S_K .
- (c) If \mathbb{L} is a linear system on S without fixed components and such that $\dim \mathbb{L} \geq 1$, let $\tilde{\mathbb{L}}^K$ denote the strict transform of \mathbb{L} on S_K .
- (d) Given $P \in K$, one can define the corresponding exceptional curve E_P as follows. Consider the subcluster $K' = \{x \in K \mid x \leq P\}$ of K and factor π_K as in (1). Then $E_P \subset S_{K'}$ is the unique irreducible component of the exceptional locus of $\pi_{K'}$ with self-intersection (-1) . The strict transform (resp. total transform) of $\pi_{K'}$ with self-intersection (-1) . The strict transform (resp. total transform) of E_P on S_K is denoted $\tilde{E}_P^K \subset S_K$ (resp. $\overline{E}_P^K \in \text{Div}(S_K)$).

- 1.2. (a) Given $P \in S^*$, consider the blowing-up $\pi_{K^P} : S_{K^P} \rightarrow S$ of S along the cluster $K^P = \{x \in S^* \mid x < P\}$, and note that P is a proper point of S_{K^P} .
- (b) Given $P \in S^*$ and a curve $C \subset S$, let $e_P(C) \in \mathbb{N}$ denote the multiplicity of C at P (by definition, this is the multiplicity of the curve $\tilde{C}^{K^P} \subset S_{K^P}$ at the proper point P of S_{K^P}). Extending linearly, let $e_P(D) \in \mathbb{Z}$ denote the multiplicity of a divisor $D \in \text{Div}(S)$ at P .

- (c) Given $P \in S^*$ and a linear system \mathbb{L} on S without fixed components and such that $\dim \mathbb{L} \geq 1$, let $e_P(\mathbb{L}) \in \mathbb{N}$ denote the multiplicity of \mathbb{L} at P (by definition, $e_P(\mathbb{L}) = \min \{ e_P(D) \mid D \in \widetilde{\mathbb{L}}^{K^P} \}$). Note that the general member D of $\widetilde{\mathbb{L}}^{K^P}$ satisfies $e_P(D) = e_P(\mathbb{L})$.

1.3. A *weighted cluster* on S is a pair (K, m) where K is a cluster on S and $m : K \rightarrow \mathbb{Z}$ is any set map. If K' is a subcluster of K and $m' : K' \rightarrow \mathbb{Z}$ is the restriction of m , we call (K', m') a *weighted subcluster* of (K, m) .

1.4. Consider an effective divisor $D \in \text{Div}(S)$.

- (a) Define the set $K^D = \{ P \in S^* \mid e_P(D) > 1 \}$ and note that this is a finite set if and only if D is reduced.
- (b) Assume that D is reduced. Then K^D is a cluster on S , called the *cluster of singular points of D* . If $K^D = \emptyset$, we say that D is nonsingular. The blowing-up $\pi_{K^D} : S_{K^D} \rightarrow S$ of S along K^D is called the *minimal resolution of singularities of D* . For an arbitrary cluster K on S ,

$$\widetilde{D}^K \text{ is nonsingular} \iff K^D \subseteq K. \quad (2)$$

- (c) Continue to assume that D is reduced. If $e(D) : K^D \rightarrow \mathbb{Z}$ denotes the map $P \mapsto e_P(D)$ then we call $\mathcal{K}^D = (K^D, e(D))$ the *weighted cluster of singular points of D* .

1.5. Consider a linear system \mathbb{L} on S such that $\dim \mathbb{L} \geq 1$ and without fixed components.

- (a) The set $K_{\mathbb{L}} = \{ P \in S^* \mid e_P(\mathbb{L}) > 0 \}$ is a cluster on S , called the *cluster of base points of \mathbb{L}* . The blowing-up $\pi_{K_{\mathbb{L}}} : S_{K_{\mathbb{L}}} \rightarrow S$ of S along $K_{\mathbb{L}}$ is called the *minimal resolution of the base points of \mathbb{L}* . For an arbitrary cluster K on S ,

$$\widetilde{\mathbb{L}}^K \text{ is base-point-free} \iff K_{\mathbb{L}} \subseteq K. \quad (3)$$

Let us also observe the following property of $K_{\mathbb{L}}$:

For each $P \in K_{\mathbb{L}}$, if $(\widetilde{E}_P^{K_{\mathbb{L}}})^2 = -1$ in $S_{K_{\mathbb{L}}}$ then $\widetilde{E}_P^{K_{\mathbb{L}}}$ is a horizontal curve (i.e., is not included in the support of an element of $\widetilde{\mathbb{L}}^{K_{\mathbb{L}}}$). (4)

- (b) If $e(\mathbb{L}) : K_{\mathbb{L}} \rightarrow \mathbb{Z}$ denotes the map $P \mapsto e_P(\mathbb{L})$ then we call $\mathcal{K}_{\mathbb{L}} = (K_{\mathbb{L}}, e(\mathbb{L}))$ the *weighted cluster of base points of \mathbb{L}* .
- (c) We write $\text{Bs}(\mathbb{L}) = \{ Q \in S \mid e_Q(\mathbb{L}) > 0 \}$ for the *base locus* of \mathbb{L} . Note that this is a set of proper points of S , and is the set of minimal elements of $K_{\mathbb{L}}$.

1.6. Let $\mathcal{K} = (K, m)$ be a weighted cluster on S and D a divisor on S . Let us use the notation $m = (m_P)_{P \in K}$ for the map m and let $\pi_K : S_K \rightarrow S$ be the blowing-up of S along K .

- (a) The *virtual transform of D with respect to \mathcal{K}* is the divisor $\check{D}^{\mathcal{K}} \in \text{Div}(S_K)$ defined by:

$$\check{D}^{\mathcal{K}} = \overline{D}^K - \sum_{P \in K} m_P \overline{E}_P^K.$$

- (b) We say that D goes through \mathcal{K} if $\check{D}^{\mathcal{K}}$ is an effective divisor. Note that if D goes through \mathcal{K} then D is effective.
- (c) We say that D goes through \mathcal{K} effectively if the following equivalent conditions are satisfied:
 - D is effective and $e_P(D) = m_P$ for all $P \in K$
 - D goes through \mathcal{K} and $e_P(D) = m_P$ for all $P \in K$
 - D is effective and $\check{D}^{\mathcal{K}} = \tilde{D}^K$.

We leave it to the reader to verify assertions 1.7–1.11, below. To prove 1.11(b), one uses characteristic zero Bertini Theorem.

1.7. Let \mathcal{K} be a weighted cluster on S and $D \in \text{Div}(S)$. If D goes through \mathcal{K} , then D goes through every weighted subcluster of \mathcal{K} .

1.8. Let $\mathcal{K} = (K, m)$ be a weighted cluster on S and $D \in \text{Div}(S)$. Suppose that D goes through \mathcal{K} , and that $e_P(D) \leq m_P$ for all $P \in K$. Then $e_P(D) = m_P$ for all $P \in K$.

1.9. Let \mathbb{L} be a linear system on S without fixed component and such that $\dim \mathbb{L} \geq 1$. For any $D \in \mathbb{L}$ and any cluster K on S , the following are equivalent:

- (a) $\tilde{D}^K \in \tilde{\mathbb{L}}^K$
- (b) $e_P(D) = e_P(\mathbb{L})$ for all $P \in K$
- (c) $e_P(D) \leq e_P(\mathbb{L})$ for all $P \in K$
- (d) D goes through the weighted cluster $(K, e(\mathbb{L}))$ effectively, where $e(\mathbb{L}) : K \rightarrow \mathbb{Z}$ denotes the set map $P \mapsto e_P(\mathbb{L})$.

1.10. **Notation.** If $\mathcal{K} = (K, m)$ is a weighted cluster, let $\mathcal{K}^{(>1)} = (K', m')$ be the pair defined by setting $K' = \{P \in K \mid m(P) > 1\}$ and by letting $m' : K' \rightarrow \mathbb{Z}$ be the restriction of $m : K \rightarrow \mathbb{Z}$ to K' .

1.11. Let \mathbb{L} be a linear system on S without fixed component and such that $\dim \mathbb{L} \geq 1$.

- (a) For any $D \in \mathbb{L}$, the following are equivalent:
 - (i) $\mathcal{K}^D = \mathcal{K}_{\mathbb{L}}^{(>1)}$
 - (ii) $\tilde{D}^{K_{\mathbb{L}}} \in \tilde{\mathbb{L}}^{K_{\mathbb{L}}}$ and $\tilde{D}^{K_{\mathbb{L}}}$ is nonsingular.
- (b) The general member D of \mathbb{L} satisfies (a-i) and (a-ii), and goes through $\mathcal{K}_{\mathbb{L}}$ effectively.

2. RATIONAL LINEAR SYSTEMS ON RATIONAL SURFACES

In this section, S is a rational nonsingular projective surface.

2.1. **Definition.** A linear system \mathbb{L} on S is *rational* if $\dim \mathbb{L} \geq 1$ and the general member of \mathbb{L} is a rational curve.

Given a curve $C \subset S$, it is interesting to ask whether there exists a rational linear system \mathbb{L} on S satisfying $C \in \mathbb{L}$. In this section we show that the existence of \mathbb{L} is equivalent to C being rational and of nonnegative type (cf. 2.5). When C satisfies these conditions, we describe all rational linear systems containing C .

We begin by recalling some known facts (2.3 and 2.4).

2.2. Definition. A pencil Λ on S is called a \mathbb{P}^1 -ruling if it is base-point-free and if its general member is isomorphic to a projective line.

The following fact is a consequence of a well-known result of Gizatullin (see for instance [5, Chap. 2, 2.2] or [3, Sec. 2]). Note that Gizatullin's result is stronger than 2.3, as we are only stating the part of the result which we need.

2.3. Lemma (Gizatullin). *Let Λ be a \mathbb{P}^1 -ruling on S and let $D \in \Lambda$.*

- (a) *Each irreducible component of D is a nonsingular rational curve.*
- (b) *If $\text{supp}(D)$ is irreducible then D is reduced.*
- (c) *If $\text{supp}(D)$ is reducible then there exists a (-1) -component Γ of $\text{supp}(D)$ which meets at most two other components of $\text{supp}(D)$; moreover, if Γ has multiplicity 1 in the divisor D then there exists another (-1) -component of $\text{supp}(D)$ which meets at most two other components of $\text{supp}(D)$.*

2.4. Lemma. *Consider $C \subset S$ such that $C \cong \mathbb{P}^1$ and $C^2 \geq 0$.*

- (a) *$\dim |C| = C^2 + 1$ and $|C|$ is base-point-free.*
- (b) *For any linear system \mathbb{L} on S such that $C \in \mathbb{L}$ and $\dim \mathbb{L} \geq 1$, the general member of \mathbb{L} is a nonsingular rational curve.*
- (c) *If $C^2 = 0$ then $|C|$ is a \mathbb{P}^1 -ruling.*

Proof. Assertions (a) and (c) are well known. Let \mathbb{L} be a linear system on S such that $C \in \mathbb{L}$ and $\dim \mathbb{L} \geq 1$, and consider a general member D of \mathbb{L} . Then D is irreducible and reduced (because \mathbb{L} has an element which is irreducible and reduced) and $p_a(D) = p_a(C) = 0$ (because D is linearly equivalent to C); so D is a nonsingular rational curve. \square

Let us now turn our attention to the subject matter of this section, i.e., the problem of describing all rational linear systems containing a given curve.

2.5. Definition. Let $C \subset S$ be a curve. Consider the minimal resolution of singularities $\pi = \pi_{KC} : S_{KC} \rightarrow S$ of C (cf. 1.4), and the strict transform $\tilde{C} = \tilde{C}^{KC} \subset S_{KC}$ of C . Let $\tilde{\nu}(C)$ denote the self-intersection number of \tilde{C} in S_{KC} . If $\tilde{\nu}(C) \geq 0$, we say that C is of *nonnegative type*. We also define the set

$$\mathbb{L}_C = \{ \pi_*(D) \mid D \in |\tilde{C}| \}$$

where $\pi_* : \text{Div}(S_{KC}) \rightarrow \text{Div}(S)$ is the homomorphism induced by $\pi = \pi_{KC}$. It is clear that \mathbb{L}_C is a linear system on S , that $\dim \mathbb{L}_C = \dim |\tilde{C}|$, and that $C \in \mathbb{L}_C$.

2.6. Lemma. *Let $C \subset S$ be a rational curve.*

- (a) $\dim \mathbb{L}_C \geq 1 \Leftrightarrow \tilde{\nu}(C) \geq 0$
- (b) *If $\tilde{\nu}(C) \geq 0$, then $\dim \mathbb{L}_C = \tilde{\nu}(C) + 1$.*
- (c) *If $\tilde{\nu}(C) \geq 0$, then every linear system \mathbb{L} on S satisfying $C \in \mathbb{L} \subseteq \mathbb{L}_C$ and $\dim \mathbb{L} \geq 1$ is a rational linear system.*

Proof. Let the notation $(\pi = \pi_{K^C} : S_{K^C} \rightarrow S, \pi_* : \text{Div}(S_{K^C}) \rightarrow \text{Div}(S))$ and $\tilde{C} = \tilde{C}^{K^C} \subset S_{K^C}$ be as in 2.5. We have $\dim \mathbb{L}_C = \dim |\tilde{C}|$ and $\tilde{\nu}(C) = \tilde{C}^2$, so assertion (b) follows by applying 2.4(a) to the nonsingular curve \tilde{C} . Part “ \Leftarrow ” of (a) follows immediately, and the converse is the observation that $\dim |\tilde{C}| \geq 1$ implies $\tilde{C}^2 \geq 0$.

To prove (c), suppose that $\tilde{\nu}(C) \geq 0$ and consider a linear system \mathbb{L} on S satisfying $C \in \mathbb{L} \subseteq \mathbb{L}_C$ and $\dim \mathbb{L} \geq 1$. Then there exists a linear system \mathbb{L}' on S_{K^C} satisfying $\tilde{C} \in \mathbb{L}'$ and $\pi_*(\mathbb{L}') = \mathbb{L}$. Since $\tilde{C}^2 = \tilde{\nu}(C) \geq 0$, 2.4(b) implies that the general member of \mathbb{L}' is a rational curve; so the general member of $\pi_*(\mathbb{L}') = \mathbb{L}$ is a rational curve. \square

2.7. Proposition. *Let $C \subset S$ be a curve and suppose that \mathbb{L} is a rational linear system on S (cf. 2.1) satisfying $C \in \mathbb{L}$. Then the following hold.*

- (a) C is a rational curve of nonnegative type.
- (b) $\tilde{C}^{K_{\mathbb{L}}} \in \tilde{\mathbb{L}}^{K_{\mathbb{L}}}$ and $\tilde{C}^{K_{\mathbb{L}}}$ is nonsingular.
- (c) C goes through $\mathcal{K}_{\mathbb{L}}$ effectively.
- (d) The general member D of \mathbb{L} satisfies $\mathcal{K}^C = \mathcal{K}^D$.
- (e) $\mathbb{L} \subseteq \mathbb{L}_C$
- (f) $K^C \subseteq K_{\mathbb{L}}$ and $\tilde{\nu}(C) = (\tilde{C}^{K_{\mathbb{L}}})^2 + |K_{\mathbb{L}} \setminus K^C| \geq |K_{\mathbb{L}} \setminus K^C|$.

Proof. There is a nonempty Zariski-open subset U of \mathbb{L} such that every element of U is an irreducible rational curve. Pick a pencil $\Lambda \subseteq \mathbb{L}$ such that $C \in \Lambda$ and $\Lambda \cap U \neq \emptyset$; then Λ is a rational pencil. Let $\pi_{K_{\Lambda}} : S_{K_{\Lambda}} \rightarrow S$ be the minimal resolution of the base points of Λ . Then $\tilde{\Lambda}^{K_{\Lambda}}$ is a \mathbb{P}^1 -ruling and $\tilde{C}^{K_{\Lambda}}$ is included in the support of an element of $\tilde{\Lambda}^{K_{\Lambda}}$, so Gizatullin’s Theorem 2.3 implies that $\tilde{C}^{K_{\Lambda}}$ is rational (so C is rational) and nonsingular (so $K^C \subseteq K_{\Lambda}$ by (2)). Let $F \in \tilde{\Lambda}^{K_{\Lambda}}$ be the element such that $\tilde{C}^{K_{\Lambda}} \subseteq \text{supp}(F)$. The fact that $1C \in \Lambda$ implies that

$$F = 1\tilde{C}^{K_{\Lambda}} + \sum_{P \in I} a_P \tilde{E}_P^{K_{\Lambda}} \quad (5)$$

for some subset $I \subseteq K_{\Lambda}$ and where $a_P \geq 1$ for all $P \in I$.

We claim that $I = \emptyset$. Indeed, suppose the contrary. Then $\text{supp}(F)$ is reducible, so Gizatullin’s Theorem implies that $\text{supp}(F)$ has a (-1) -component Γ , and that if Γ has multiplicity 1 in F then Γ is not the only (-1) -component of $\text{supp}(F)$. This together with (5) imply that there exists $P \in I$ such that $(\tilde{E}_P^{K_{\Lambda}})^2 = -1$; as $P \in K_{\Lambda}$ and $\tilde{E}_P^{K_{\Lambda}}$ is vertical, this contradicts (4), and proves that $I = \emptyset$. So:

$$\tilde{C}^{K_{\Lambda}} \in \tilde{\Lambda}^{K_{\Lambda}}. \quad (6)$$

It follows that $(\tilde{C}^{K_{\Lambda}})^2 = 0$ in $S_{K_{\Lambda}}$, because $\tilde{\Lambda}^{K_{\Lambda}}$ is a base-point-free pencil. As $\tilde{C}^{K_{\Lambda}}$ is also nonsingular, C is of nonnegative type and (a) is proved.

Since $\tilde{C}^{K_{\Lambda}} \in \tilde{\Lambda}^{K_{\Lambda}}$ and $\tilde{C}^{K_{\Lambda}}$ is nonsingular, 1.11 implies that $\mathcal{K}^C = \mathcal{K}_{\Lambda}^{(>1)}$. As the general member D of Λ satisfies $\mathcal{K}^D = \mathcal{K}_{\Lambda}^{(>1)}$ by 1.11, we get $\mathcal{K}^D = \mathcal{K}^C$. So we have shown that, for any pencil Λ satisfying $C \in \Lambda \subseteq \mathbb{L}$ and $\Lambda \cap U \neq \emptyset$, the general member

D of Λ satisfies $\mathcal{K}^D = \mathcal{K}^C$. Consequently, $\{D \in \mathbb{L} \mid \mathcal{K}^D = \mathcal{K}^C\}$ is a dense subset of \mathbb{L} ; together with the fact (1.11) that $\mathcal{K}^D = \mathcal{K}_{\mathbb{L}}^{(>1)}$ for general $D \in \mathbb{L}$, this implies

$$\mathcal{K}^C = \mathcal{K}_{\mathbb{L}}^{(>1)}. \quad (7)$$

Then assertions (b) and (d) follow from (7) and 1.11, and assertion (c) follows from $\tilde{C}^{K_{\mathbb{L}}} \in \tilde{\mathbb{L}}^{K_{\mathbb{L}}}$ and 1.9. By (7) we have $e_P(C) = e_P(\mathbb{L})$ for all $P \in K^C$; this together with 1.9 implies that $\tilde{C}^{K^C} \in \tilde{\mathbb{L}}^{K^C}$, hence $\tilde{\mathbb{L}}^{K^C} \subseteq |\tilde{C}^{K^C}|$. It follows that $\pi_*(\tilde{\mathbb{L}}^{K^C}) \subseteq \pi_*(|\tilde{C}^{K^C}|)$, where $\pi_* : \text{Div}(S_{K^C}) \rightarrow \text{Div}(S)$ is the homomorphism induced by $\pi = \pi_{K^C} : S_{K^C} \rightarrow S$. As $\pi_*(\tilde{\mathbb{L}}^{K^C}) = \mathbb{L}$ and (by definition) $\pi_*(|\tilde{C}^{K^C}|) = \mathbb{L}_C$, (e) is true.

We have $K^C \subseteq K_{\mathbb{L}}$ by (7), and (c) implies that $e_P(C) = 1$ for all $P \in K_{\mathbb{L}} \setminus K^C$. Consequently, $\tilde{\nu}(C) = (\tilde{C}^{K^C})^2 = (\tilde{C}^{K_{\mathbb{L}}})^2 + |K_{\mathbb{L}} \setminus K^C|$. Pick any $D \in \tilde{\mathbb{L}}^{K_{\mathbb{L}}} \setminus \{\tilde{C}^{K_{\mathbb{L}}}\}$, then $(\tilde{C}^{K_{\mathbb{L}}})^2 = \tilde{C}^{K_{\mathbb{L}}} \cdot D \geq 0$ and (f) is proved. \square

2.8. Theorem. *For a curve $C \subset S$, the following are equivalent:*

- (a) \mathbb{L}_C is a rational linear system;
- (b) there exists a rational linear system \mathbb{L} on S such that $C \in \mathbb{L}$;
- (c) C is rational and $\dim \mathbb{L}_C \geq 1$;
- (d) C is rational and $\tilde{\nu}(C) \geq 0$.

Moreover, if conditions (a–d) are satisfied then the following hold:

- (e) For a linear system \mathbb{L} on S satisfying $C \in \mathbb{L}$ and $\dim \mathbb{L} \geq 1$,

$$\mathbb{L} \text{ is rational} \iff \mathbb{L} \subseteq \mathbb{L}_C.$$

- (f) $\dim \mathbb{L}_C = \tilde{\nu}(C) + 1$, $\mathcal{K}_{\mathbb{L}_C} = \mathcal{K}^C$ and $\tilde{\mathbb{L}}_C^{K^C} = |\tilde{C}^{K^C}|$.

Proof. Suppose that \mathbb{L} is a rational linear system on S such that $C \in \mathbb{L}$. By 2.7, we obtain that C is rational and that $\mathbb{L} \subseteq \mathbb{L}_C$ (so $\dim \mathbb{L}_C \geq 1$). So (b) implies (c), and this also proves implication “ \Rightarrow ” in statement (e).

Equivalence (c) \Leftrightarrow (d) is 2.6(a), implication (c and d) \Rightarrow (a) is the case $\mathbb{L} = \mathbb{L}_C$ of 2.6(c), and (a) \Rightarrow (b) is obvious. So (a–d) are equivalent.

Now assume that (a–d) are satisfied. Implication “ \Leftarrow ” in statement (e) is a consequence of 2.6(c), so there only remains to prove (f). Equality $\dim \mathbb{L}_C = \tilde{\nu}(C) + 1$ is 2.6(b). Observe that there can be at most one linear system \mathbb{G} on S_{K^C} satisfying

$$\text{the general member of } \mathbb{G} \text{ is irreducible and } \pi_*(\mathbb{G}) = \mathbb{L}_C,$$

where $\pi = \pi_{K^C} : S_{K^C} \rightarrow S$; as $\tilde{\mathbb{L}}_C^{K^C}$ and $|\tilde{C}^{K^C}|$ are two such linear systems, we get $\tilde{\mathbb{L}}_C^{K^C} = |\tilde{C}^{K^C}|$. This implies that $\tilde{\mathbb{L}}_C^{K^C}$ is base-point-free (because $|\tilde{C}^{K^C}|$ is base-point-free by 2.4), so all base points of \mathbb{L}_C are in K^C , i.e., $K_{\mathbb{L}_C} \subseteq K^C$. On the other hand, 2.7(b) together with 1.11 gives $\mathcal{K}^C = \mathcal{K}_{\mathbb{L}_C}^{(>1)}$ (which was also noted in (7)); this and $K_{\mathbb{L}_C} \subseteq K^C$ imply $\mathcal{K}_{\mathbb{L}_C} = \mathcal{K}^C$, which completes the proof. \square

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